1. Four. řada fce $f(x)$, 2π-periodické v R,

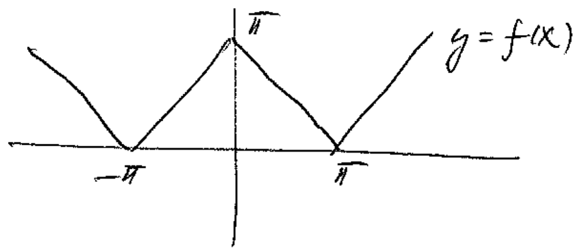
$f(x) = \pi - |x|$ pro $x \in \langle -\pi, \pi \rangle$

1) f je spojitá, 2π periodická, sudá $\Rightarrow$ F. řada $\rightrightarrows f$ na R

a F řada bude kosinová řada

$(b_n = 0,\ n=1,2 \dots)$

a $\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx = f(x)$ v R



2) „Výpočet" F. řady

$b_n = 0$ (f je sudá), $n=1,2\dots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |x|)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |x|)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx = \left| \begin{array}{ll} u' = \cos nx & u = \frac{\sin nx}{n} \\ v = x & v' = 1 \end{array} \right|$$

$$= \frac{2}{\pi}\left\{ \underbrace{\left[\pi \frac{\sin nx}{n}\right]_0^{\pi}}_{=0} - \underbrace{\left[x\cdot\frac{\sin nx}{n}\right]_0^{\pi}}_{=0} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx \right\} =$$

$$= \frac{2}{\pi}\,\frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{1}{n^2}\left(1-(-1)^n\right) = \begin{cases} 0, & n\text{-sudé} \\ \frac{4}{\pi}\cdot\frac{1}{n^2}, & n\text{-liché}, \end{cases}$$

tedy, F. ř. je $\frac{\pi}{2} + \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$ $\left(= \pi - |x| \text{ pro } x \in \langle -\pi, \pi \rangle\right)$

spec. $x=0$ $\quad \frac{\pi}{2} + \frac{4}{\pi}\sum\frac{1}{(2k+1)^2} = \pi \Rightarrow \sum\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$

2. f – 2π periodická funkce, $f(x) = \sin|2x|$, $x \in \langle -\pi, \pi)$:

(i) $f(-\pi) = \lim_{x \to \pi-} f(x) = 0 \Rightarrow$ f je spojitá fce v R (a 2π periodická)

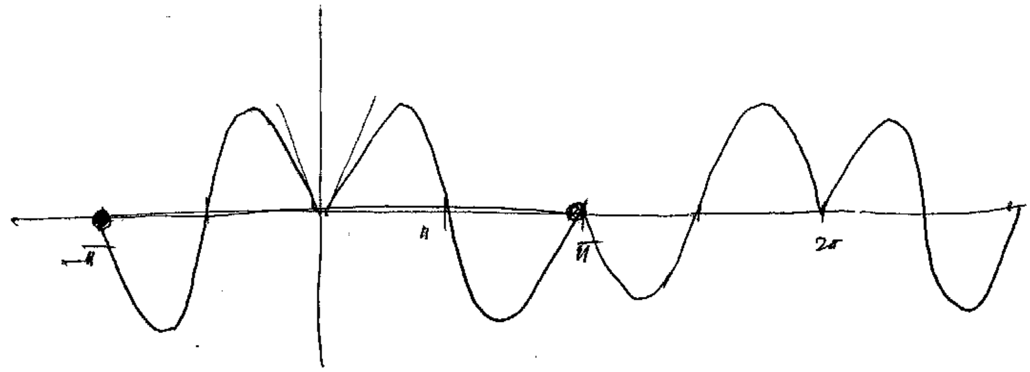
$$\left.\begin{array}{ll} f'(x) = 2\cos 2x & v\ (0,\pi) \\ f'(x) = -2\cos 2x & v\ (-\pi, 0) \end{array}\right\} \Rightarrow$$

$f'(0)$ neex. $(f'(0-) = -2,\ f'(0+) = 2)$

$f'(\pi)$ neex. $(f'(\pi-) = 2,\ f'(\pi+) = -2) \Rightarrow$ f je po částech hladká,

tedy Fourierova řada konverguje stejnoměrně v R k funkci f

graf funkce



Fourierova řada

f - sudá $\Rightarrow b_n = 0$, $\quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin|2x|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin 2x\,dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin|2x| \cdot \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(2x) \cdot \cos(nx)\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin(2+n)x + \sin(2-n)x\right)dx = \begin{cases} = 0, & n\text{-sudé} \\ & (\text{pro } n\text{-liché}) \end{cases}$$

$$= -\frac{1}{\pi}\left[\frac{\cos(2+n)x}{2+n} + \frac{\cos(2-n)x}{2-n}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{(-1)^n - 1}{2+n} + \frac{(-1)^n - 1}{2-n}\right]$$

$$\frac{2}{\pi}\,\frac{2-n+2+n}{4-n^2} = \frac{-8}{\pi}\,\frac{1}{n^2-4}, \text{ tedy:}$$

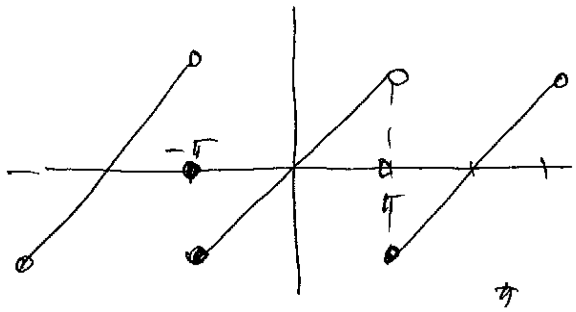
F.ř. $\quad \boxed{f(x) = \sum_{k=0}^{\infty} \frac{-8}{\pi\left[(2k+1)^2 - 4\right]} \cdot \cos(2k+1)x, \quad x \in R}$

3. $f(x)$ – 2π periodická funkce, def. v $\mathbb{R}$, $f(x) = x$ na $\langle -\pi, \pi)$

(i) f je po částech hladká v $\langle -\pi, \pi)$, f není spojitá v bodech $x = k\pi$, $k \in \mathbb{Z}$, tedy její Fourierova řada konverguje bodově ke funkci $\tilde{f}(x)$,

$$\tilde{f}(x) = \begin{cases} f(x) & \text{pro } x \neq k\pi,\ k \in \mathbb{Z} \\ 0 & \text{pro } x = k\pi \end{cases} \quad \left(= \frac{f(k\pi+0) + f(k\pi-0)}{2} \right)$$

(Dirichletova věta)



(ii) Fourierova řada f:

f je lichá funkce $\Rightarrow a_0, a_1, \dots = 0$,

$$\left(a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\,dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\,dx \right)$$

Výpočet

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\,dx \underset{\text{sudá fce}}{=} \frac{2}{\pi} \int_0^{\pi} x \sin nx\,dx \underset{PP}{=} \left| \begin{array}{l} u' = \sin nx,\ u = \frac{-\cos nx}{n} \\ v = x,\ v' = 1 \end{array} \right|$$

$$= \frac{2}{\pi} \left\{ \left[-x \frac{\cos nx}{n} \right]_0^{\pi} - \int_0^{\pi} - \frac{\cos nx}{n}\,dx \right\} =$$

$$= \frac{2}{\pi} \left\{ \left[-x \frac{\cos nx}{n} \right]_0^{\pi} + \underbrace{\left[\frac{\sin nx}{n^2} \right]_0^{\pi}}_{=0} \right\} = -\frac{2}{\pi} \frac{\pi}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

tj. $$\Phi_f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad \left(= \begin{cases} f(x) & \text{pro } x \neq k\pi \\ 0 & \text{pro } x = k\pi \end{cases} \right)$$

(iii) $x = \frac{\pi}{2}$: $\sin\left(n \frac{\pi}{2}\right) = \begin{cases} 0 & \text{pro } n \text{ sudé} \\ (-1)^k & \text{pro } n = 2k+1 \end{cases}$, tedy

$$\left.\begin{array}{l} \Phi_f\left(\frac{\pi}{2}\right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^{2k+2}}{2k+1} (-1)^k = 2 \sum \frac{(-1)^k}{2k+1} \\ \Phi\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad (f \text{ je spojitá v } x = \frac{\pi}{2}) \end{array}\right\} \Rightarrow \sum_{0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$