MAI 2 – cvičení 5

Další substituce, vedoucí k integraci racionální funkce:

d) $\int R(x, \sqrt{ax^2+bx+c})\,dx$:

vhodné substituce: $a>0$ $\quad \sqrt{ax^2+bx+c} = \pm\sqrt{a}\cdot x \pm t$ $\Big\}$ Eulerovy

$c>0$ $\quad \sqrt{ax^2+bx+c} = \sqrt{c} + x\cdot t$

$a<0$ (ax^2+bx+c má dva různé reálné kořeny $\alpha_1 < \alpha_2$):

$$\sqrt{ax^2+bx+c} = \sqrt{-a}\,(x-\alpha_1)\sqrt{\frac{\alpha_2-x}{x-\alpha_1}}\text{, nebo}$$

$$\sqrt{ax^2+bx+c} = \sqrt{-a}\,(\alpha_2-x)\sqrt{\frac{x-\alpha_1}{\alpha_2-x}} \text{ – subst. c)}$$

a další vhodné substituce (viz dole)

Příklady:

$a>0$: 1. $\int \frac{1}{\sqrt{1+x^2}}\,dx$; 2. $\int \sqrt{1+x^2}\,dx$ (také $x = \sinh t \left(= \frac{e^t - e^{-t}}{2}\right)$)

3. $\int \frac{1}{x\sqrt{1+x^2}}\,dx$ (ale také $t = \frac{1}{x}$, $1+x^2 = t$, $\sqrt{1+x^2} = t$, $x = \sinh t$)

4. $\int \frac{1}{\sqrt{x^2-1}}\,dx$ (také $x = \cosh t$, $\cosh t = \frac{e^t + e^{-t}}{2}$; $\cosh^2 t - \sinh^2 t = 1$)

5. $\int \frac{1}{x\sqrt{x^2-1}}\,dx$; 6. $\int \frac{\sqrt{1+x^2}}{x}$; 7. $\int \frac{1}{\sqrt{x^2+x+1}}\,dx$;

8. $\int \frac{1}{\sqrt{(x^2+x+1)^3}}\,dx$; 9. $\int \frac{1}{x\sqrt{x^2+x+1}}\,dx$ (ale také $\frac{1}{x} = t$);

10. $\int \frac{1}{x+\sqrt{x^2+x+1}}\,dx$; 11. $\int \frac{1}{\sqrt{x^2+2x+3}}\,dx$; 12. $\int \frac{1}{1+\sqrt{x^2+2x+2}}\,dx$

$a<0$: 13. $\int \frac{1}{\sqrt{2+x-x^2}}\,dx$; 14. $\int \frac{1}{x\sqrt{2+x-x^2}}\,dx$; 15. $\int \frac{1}{x+\sqrt{2+x-x^2}}\,dx$;

16. $\int \frac{1}{\sqrt{6+x-x^2}}\,dx$; 17. $\int \frac{x}{\sqrt{6+x-x^2}}\,dx$; 18. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$;

19. $\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$; 20. $\int \frac{1}{x+\sqrt{3+2x-x^2}}\,dx$;

Ještě perlička: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} \arcsin x + c, \quad x \in (-1,1) \\ \int \frac{1}{x+1}\cdot\sqrt{\frac{1+x}{1-x}}\,dx \end{cases}$.

e) $\int R(\sin x, \cos x)\,dx$:

(i) substituce $\sin x = t$ nebo $\cos x = t$:

1. $\int \frac{\cos x}{\sin^3 x}\,dx$; 2. $\int \frac{\sin x}{2+\cos x}\,dx$; 3. $\int \frac{\sin x}{(1-\cos x)^2}\,dx$; 4. $\int \sin^5 x\,dx$;

5. $\int \sin^3 x\cdot\cos^2 x\,dx$; 6. $\int \frac{1}{\sin x}\,dx$; 7. $\int \frac{1}{\cos^3 x}\,dx$; 8. $\int \frac{1}{(2+\cos x)\sin x}\,dx$

(ii) substituce $\operatorname{tg} x = t$:

9. $\int \frac{1}{1+\operatorname{tg} x}\,dx$; 10. $\int \frac{1}{(\operatorname{tg}^2 x+1)^2}\cdot\frac{1}{\cos^2 x}\,dx$; 11. $\int \frac{1}{\sin^2 x\cdot\cos^2 x}\,dx$;

12. $\int \frac{\sin^2 x}{1+\sin^2 x}\,dx$; 13. $\int \frac{1}{(\sin x+\cos x)^2}\,dx$

(iii) substituce $\operatorname{tg}\frac{x}{2} = t$:

14. $\int \frac{1}{2+\cos x}\,dx$; 15. $\int \frac{dx}{5+4\sin x}\,dx$; 16. $\int \frac{2+\sin x}{2-\sin x}\,dx$.